Stratified disks are locally stable

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Abstract. Notwithstanding recent claims by Richard *et al.*, there is no linear hydrodynamic instability of axisymmetrically stable disks in the local limit. We prove this by means of an exact stability analysis of an unbounded incompressible flow having constant stratification and constant shear.

Key words. Accretion disks – hydrodynamic instabilities – turbulence

1. Introduction

In a very recent preprint, Richard et al. (2001) claim to demonstrate nonaxisymmetric local incompressible linear instability of shearing disks with vertical stratification. (They also discuss more general flows that are stratified in both radius and height). The growth rate is approximately the geometric mean of the shear rate (S) and Brunt-Väisälä frequency (N), hence quite rapid since $|S/\Omega| = 3/2$ in keplerian disks, and typically one expects $N \sim \Omega$ away from the disk midplane. If true, this would be very important, as the instability might support turbulence and angular-momentum transport in disks too weakly ionized for magnetorotational instabilities (Balbus & Hawley, 1998).

In §2, we derive the exact equation (14) for time-evolution of spatial Fourier modes in a Boussinesq flow with a linear shear profile. In §3, we use this equation to show that all local disturbances are bounded at late times, and hence there is no growing local mode. We also briefly discuss how global instabilities might arise in the presence of reflecting boundaries.

2. Local Boussinesq model

Using a local WKB analysis, Richard et al. (2001) assert that local Fourier modes with wavenumber $\boldsymbol{k} = (k_r, k_\theta, k_z)$ grow $\propto \exp(\Gamma t)$ [1]

$$\Gamma^2 = -\frac{2k_\theta^2 N^2 \Omega S}{k_\theta^2(N^2 + 2\Omega S) + k_r^2 N^2 + k_z^2 \kappa^2}. \qquad (1)$$

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[1] The term $k_r^2 N^2$ in the denominator is missing in the expression quoted in Richard et al. (2001). This appears to be a typographical error. Whether the term is present or not does not affect our critique.

As usual, $r\Omega(r)\boldsymbol{e}_\theta$ is the background orbital velocity, $S = d\Omega/d\ln r < 0$ is the radial shear, and

$$\kappa^2 = r^{-3}\frac{d(r^4\Omega^2)}{dr} = 2\Omega(2\Omega + S),$$

$$N^2 = -\frac{1}{\gamma\rho}\frac{\partial P}{\partial z}\frac{\partial}{\partial z}\ln\left(\frac{P}{\rho^\gamma}\right)$$

are the squares of the epicyclic and Brunt-Väisälä frequencies. Since it is assumed that $N^2 > 0$ and $\kappa^2 > 0$, the flow would be stable against both convection and centrifugal instability if there were no shear. Yet according to eq. (1), there is a constant growth rate $\omega \sim \sqrt{-2\Omega S}$ in the limit of strong stratification $N^2 \to \infty$ provided only that $d(\Omega^2)/dr < 0$ (as is normally the case in astrophysical disks).

In the extreme local limit, one considers disturbances on scales very small compared to the distance to the nearest boundary of the flow. Also, provided that Ω and N^2 are smooth functions, they vary little on the small length-scale. It is therefore reasonable to consider an unbounded flow in which all of the frequencies Ω, N, S, κ are constant in space and time. Variations in density can be ignored except in the buoyancy term (Boussinesq approximation). The governing equations for this flow in a corotating frame are

$$0 = \partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + 2\Omega\boldsymbol{e}_z\times\boldsymbol{v} + \boldsymbol{\nabla}\psi + N^2 L\,\vartheta\boldsymbol{e}_z,$$

$$\frac{v_z}{L} = \partial_t\vartheta + \boldsymbol{v}\cdot\boldsymbol{\nabla}\vartheta,$$

$$0 = \boldsymbol{\nabla}\cdot\boldsymbol{v}. \qquad (2)$$

The quantity ϑ represents the fractional density difference between a displaced fluid element and the background (which the Boussinesq approximation considers to be infinitesimal). L is a constant scale height for the entropy, so

that $N^2 = g/L$ if g is the vertical gravitational acceleration. ψ is a scalar potential for accelerations due to gravity and pressure. The background state is

$$\boldsymbol{v}_0 = Sx\boldsymbol{e}_y, \quad \vartheta_0 = 0, \qquad \psi_0 = \Omega Sx^2. \tag{3}$$

The coordinates r, θ, z have been replaced by local Cartesians x, y, z. Were Richard et al. (2001)'s analysis correct, their growth rate (1) would apply to this model.

At any given time (t), linearized disturbances of this flow can be be decomposed into spatial Fourier components,

$$\boldsymbol{v}_1 = \epsilon\boldsymbol{V}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \quad \vartheta_1 = \epsilon\Theta e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \quad \psi_1 = \epsilon\Psi e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{4}$$

where $\epsilon \ll 1$ is a formal small parameter. Following a method attributed to Kelvin (1887), (4) is a solution to the linearized equations of motion for *all* time if $\boldsymbol{V}$, Θ, Ψ *and* $\boldsymbol{k}$ are appropriate functions of t [but independent of position $\boldsymbol{r} = (x, y, z)$]. Note that this means that the time dependence of the flow attributes will not, in general, be a simple exponential function, in contrast to the Richard et al. (2001) assumption. Substituting into the first of eqs. (2) and collecting terms of first order in ϵ,

$$
\begin{aligned}
0 = {} & \left(\frac{d\boldsymbol{k}}{dt}\cdot\boldsymbol{r} + Sxk_y\right)i\boldsymbol{V} \\
& + \left(\frac{d\boldsymbol{V}}{dt} + SV_x\boldsymbol{e}_y + 2\Omega\boldsymbol{e}_z\times\boldsymbol{V} + i\boldsymbol{k}\Psi + N^2L\Theta\boldsymbol{e}_z\right).
\end{aligned}
$$

The coefficients of each coordinate xyz must vanish separately, whence

$$\frac{dk_x}{dt} = -Sk_y, \quad \frac{dk_y}{dt} = \frac{dk_z}{dt} = 0, \tag{5}$$

and

$$\frac{d\boldsymbol{V}}{dt} + SV_x\boldsymbol{e}_y + 2\Omega\boldsymbol{e}_z\times\boldsymbol{V} + i\boldsymbol{k}\Psi + N^2L\Theta\boldsymbol{e}_z = 0. \tag{6}$$

By similar steps,

$$\frac{d\Theta}{dt} = \frac{V_z}{L} \tag{7}$$

and

$$\boldsymbol{k}\cdot\boldsymbol{V} = 0 \iff V_z = -\frac{k_xV_x + k_yV_y}{k_z}. \tag{8}$$

Combining the time derivative of (8) with eq. (6) yields

$$\Psi = \frac{i}{k^2}\left[2(S+\Omega)k_yV_x - 2\Omega k_xV_y + N^2Lk_z\Theta\right]. \tag{9}$$

Note the abbreviations

$$k^2 = k_x^2 + k_y^2 + k_z^2, \qquad k_\perp^2 = k_x^2 + k_y^2 = k^2 - k_z^2. \tag{10}$$

Since V_z can be eliminated *via* eq. (8), the system (6)-(7) is third order in time.[2] However, it has a first integral,

$$k_xU_y - k_yU_x - (2\Omega + S)k_zL\Theta = \text{constant} \tag{11}$$

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[2] Yet Richard et al. (2001)'s dispersion relation is fourth-order in frequency. At $k_y = 0$, two roots vanish, and the claim is made that these develop into $\pm\Gamma$ as given by (1) when $k_y \neq 0$. Non-vortical perturbations, however, are restricted to obey a *second* order system. Moreover, if time dependences were in fact exponential, then a nonzero value of the vorticity constant in (11) would correspond to a zero eigenfrequency at *any* k_y.

representing conservation of potential vorticity. For growing modes, the constant would be asymptotically negligible. So we set it to zero. Eliminating V_z between eqs. (7) and (8):

$$k_xU_x + k_yU_y = -k_zL\frac{d\Theta}{dt}. \tag{12}$$

We use eqs. (11) & (12) to eliminate U_x and U_y from eq. (6) in favor of Θ & $\dot{\Theta}$:

$$
\begin{aligned}
0 = {} & \frac{d^2\Theta}{dt^2} + \left(\frac{2Sk_xk_yk_z^2}{k^2k_\perp^2}\right)\frac{d\Theta}{dt} \\
& + \left[N^2\frac{k_\perp^2}{k^2} + \kappa^2\frac{k_z^2}{k^2} + 2S(2\Omega+S)\frac{k_y^2k_z^2}{k^2k_\perp^2}\right]\Theta. \tag{13}
\end{aligned}
$$

(Note that eq. (13) also describes the evolution of vertical Lagrangian displacements, a consequence of eq. (7).) The variable

$$\Phi = \frac{k}{k_\perp}\Theta$$

obeys a slightly tidier equation:

$$
\begin{aligned}
0 = {} & \frac{d^2\Phi}{dt^2} + \omega^2(t)\Phi, \\
\omega^2(t) = {} & N^2\frac{k_\perp^2}{k^2} + \kappa^2\frac{k_z^2}{k^2} \\
& + \frac{k_y^2k_z^2}{k^2k_\perp^2}\left(4\Omega S - \frac{k_x^2}{k^2}S^2 + \frac{3k_y^2}{k_\perp^2}S^2\right). \tag{14}
\end{aligned}
$$

Together with the solution of eq. (5),

$$k_x(t) = k_x(0) - 2Stk_y, \qquad k_y, k_z = \text{constants}, \tag{15}$$

eq. (14) governs the evolution of a Fourier mode for the density perturbation. The equation is exact within the context of the model defined by eqs. (2)-(3). In fact, it is not even restricted to small amplitudes, because the terms quadratic in ϵ vanish when eq. (9) is substituted into eq. (2) as a consequence of eq. (8).

3. Discussion

Clearly, eq. (1) is incompatible with eq. (14). When $k_z = 0$, for example, eq. (1) predicts a finite growth rate. The exact solution of eq. (14) corresponds to vertical oscillations at the Brunt-Väisälä frequency, an elementary result (consider purely vertical displacements) that may be obtained directly from the z component of equation (6).

More generally, as $St \to \pm\infty$, $k_\perp^2 \approx k^2 \approx k_x^2 \gg k_y^2, k_z^2$, so that $\omega^2(t) \to N^2$ and $\Phi \propto \exp(\pm iNt)$. Hence, contrary to the result claimed by Richard et al. (2001) [eq. (1)], *there is no exponentially growing linear mode.*[3]

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[3] This assumes that the disk is stably stratified. On the contrary, when $N^2 < 0$, an exponentially growing convective disturbance persists as $k_x \to \infty$ if, as here, viscosity and thermal conduction are ignored (*e.g.,* Ryu & Goodman, 1992).

An approximate dispersion relation can be read off from eq. (14) in the limit $k_x \gg k_y, k_z$ (*i.e.* at late times). The corresponding radial group velocity is

$$\frac{\partial \omega}{\partial k_x} \approx \mp \frac{S(N^2 - \kappa^2)}{N} \frac{k_z^2 k_y}{k_x^3} \qquad (16)$$

This tends to zero as t^{-3}. It can be shown that a radially localized wavepacket, obtained by superposing Fourier modes with a range of $k_x(0)$, asymptotically approaches resonant radii where the frequency of the disturbance $= \pm N$ in a frame corotating with the local background. If the disturbance is excited with azimuthal wavenumber k_y by a force corotating steadily at radius r_0, the resonances lie approximately at $r_0 \pm N/Sk_y$. In order that our local analysis apply to radially bounded flows, it is necessary that these resonant radii lie inside the fluid. Otherwise the wavepacket may reflect from the boundary, suffering a change in the sign of k_x, and offering the possibility of repeated passages through the "swing amplifier" at small k_x/k_y where ω^2 can be briefly negative.[4]

The instability discussed by by Molemaker, McWilliams, & Yavneh (2001) for stratified, centrifugally-stable Couette flow, and cited by Richard et al. (2001) as support for their claims of instability, is an intrinsically global phenomenon, and has no local counterpart. Molemaker, McWilliams, & Yavneh (2001) analyzed their instability in terms of trapped Kelvin wave edge modes; it is also possible that the resonant interaction described above may be involved in destabilizing some of the modes. In either case, the mechanism is certainly not reducible to exponentially-growing local Brunt-Väisälä oscillations. A full analysis of the global problem is beyond the scope of this note.

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[4] See Toomre (1981) and Goldreich & Tremaine (1978) for particularly clear discussions of the swing-amplifier in the context of two-dimensional compressible disks.